A First Course in Partial Differential Equations

August 21, 2014

1 Preface

The study of partial differential equations (pdes) has been around since the 18th century, and it is one of the largest subdisciplines of mathematics. Its roots are in applications and in mathematical modeling of physical phenomenon. Our basic physical laws are codified in pdes. Examples come from fluid mechanics (e.g. the Navier-Stokes equations, lubrication theory), solid mechanics (e.g. beam and plate equations), electrodynamics (e.g. Maxwells equations), heat and mass transfer (e.g. diffusion equation), genetics and population dynamics (e.g. Fishers equation and more general reaction-diffusion equations), neurobiology (e.g. Hodgkin-Huxley equations), financial engineering (e.g. Black-Scholes equation), etc. PDEs have helped advance areas like probability theory, complex analysis, differential equations, Lie algebra theory, to name a few core mathematical areas.

This introductory text covers a small, but important part, of the subject, namely exact methods for solving a few classical problems in some restricted settings. We concentrate on some linear pdes where we can find explicit solutions. We address some questions in a qualitative manner from time to time, but this is not a theory course. It is mainly an analytic course of finding explicit representations for solutions. Where theorems are presented, they are meant to summarize a concept or properties of a particular subject. *To get the most out of this course of study it is expected that you learn and understand the content of theorems*, both to give some mental structure to the course, and understand why we do certain procedures the way we do them.

That said, students often find introductory partial differential equations difficult. The main reason is that you must be *really comfortable* with basic algebra, calculus, and parts of elementary differential equations. Appendices A and B are included to have you review basic prerequisite material before you get involved in the details of pde methods. The pde concepts are not hard, but there are a lot of little details you must have at your command, and you must be proficient in your background to obtain a real working understanding of the subject. *You must do as many problems as your time allows!* "She/he who does the most problems wins." This means that you do the exercises buried in the notes, and not just do assigned homework problems.

To put some perspective on the subject matter discussed in this course, keep in mind that a lot of physical modeling arise as nonlinear pdes, or systems of such equations. Even when a model problem arises that has a linear, scalar equation, it is often spatially multi-dimensional. Our study of pdes in this text is mostly about linear, scalar, pdes of one space variable. We must learn to "walk before we can run." You need to get a firm understanding of the solution approach in an analytically simpler case. But many techniques we discuss in the course are readily extended to the spatially multidimensional case. There are plenty of problems around of major physical significance that our methodologies are directly applicable to, but also keep in mind that most differential equations, ordinary and partial, are not solvable in closed-form.

However, exact (closed-form) solutions of differential equations plays an important role in the proper understanding of qualitative features of many phenomena and processes, and allow understanding of the mechanisms that give rise to such observed behavior as shock waves, energy transfer, pattern formation, traveling waves, bifurcation phenomena, phase transitions, stability and vibration phenomena, etc. Even those special exact solutions that do not have a clear physical meaning can be used as "test problems" to verify the consistency, and estimate errors, of various numerical and approximate analytical methods.

2 "Strategic Principles"

There are two overriding "principles" we adhere to in the development of this pde course as presented in these Notes.

Principle 1: We will (almost) always boil down our solution approach by reducing the partial differential equation problem to solving ordinary differential equation (ode) problems.

This is why you must become good, and quick, about solving ode problems.
Example: For Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ we approach this equation by separating variables, assuming the equation is defined on a bounded (e.g. rectangular) domain. This entails writing $u(x, y) = f(x)g(y)$ and substituting this form into the pde to obtain odes for the single variable functions f and g.

Example: You'll find out that a solution form for the heat equation $\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = 0$ for $-\infty < x < \infty$ can be given by a function of a single variable, that is $u(x, t) = f(z)$, where $z = \frac{x}{\sqrt{Dt}}$. Substituting this representation into the heat equation gives, after using the chain rule several times, $\frac{d^2 f}{dz^2} + \frac{z}{2}\frac{df}{dz} = 0$; that is, deriving a general solution means solving an ode. (In this case we assume a spatially unbounded domain.)

Principle 2: When presented with a new problem, we try to reduce it to a problem we have already solved.

Example: We go through and solve the heat equation $\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = 0$ in various situations. But what if we are faced with the diffusion-decay equation $\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} + ku = 0$? We can write $u(x, t) = e^{-kt}w(x, t)$ and show the w solves the heat equation $\frac{\partial w}{\partial t} - D\frac{\partial^2 w}{\partial x^2} = 0$. Since we have worked with this equation previously, finding w implies we have solved for u because they differ only by an exponential factor.

3 Remarks to Instructors on the Structure of these Notes

There are now a number of excellent books on the introduction of partial differential equations, so why consider these Notes?

The author has tried various textbooks in courses, but often the students found the material too hard or dense, as presented, or not enough examples of worked problems. We have tried to gear the level of exposition in these

Notes to accommodate the backgrounds of our students (to a typical state residential research university). We have also often provided exercises right after discussion of a topic, and not just put exercises at the end of a chapter. Hopefully this aids self-study of the material. While not getting carried away with a large number of exercises, we have tried a mix of standard and not so standard exercises, with the belief that students should do all the exercises in the Notes for maximum comprehension of the material.

Most undergraduate mathematics programs have a single semester or quarter course in introductory pdes, rather that a year sequence. Textbooks are getting bulkier (and very expensive) introducing a year or more worth of material. To hold costs down, we wanted a single semester's worth of core material, based on many years experience teaching such a course. That also meant we wanted to organize the core material in a way that suited our pedagogical approach to the subject. This, of course, forces compromises as to what is included in the course, and what is excluded, and not all instructors are going to agree with the author's decisions on this matter. However, instructors are always free to supplement any text material with their own material. We have included a sequence of appendices on topics not elaborated on in the main body of the Notes, but the 24 chapters of the main text make an appropriate, if intense, 15 week course.

One particular topic omission concerns numerical methods for pdes. While this is an important topic, there is no way to include a meaningful discussion without draconian cuts to core analytical material. Students should get a good understanding of solution behavior of some elementary pdes and how to transform some problems to other problems. But students should be encourage to take a follow-up course where they get experience solving pdes numerically.

A main point in the author developing these notes, however, is the opportunity to structure the content in a more appealing way. This is, of course, a matter of personal taste, but let us make a defense of the sequence of topics, and what we consider is important material to have in this first course.

First, we do not believe a first course in pdes should be just a separation-of-variables course. This drives us to first consider problems with no finite spatial boundaries (initial-value problems) and to interpret solution behavior without interference of boundary affects. This also allows us to discuss important transformations, including the Fourier transform. Later we add a single boundary, which allows us to use the Laplace transform, a topic students should had been introduced to in their prerequisite ode course. Then we turn to more standard boundary-value problems for the heat, wave, Laplace and Poisson equations. For the initial-value problems (Cauchy problems) we also compare solution behavior between heat and wave solutions in a summary way before introducing boundaries, again trying to balance work on solving techniques with interpretation of solution characteristics for different equations.

Since the classes of equations discussed in this course are important, but limited, there is a need to give some indication how they arise. We consider this very important, and we tie formulation of various equations together through use of the conservation of mass principle. This goes together with constant reminder that pdes are associated with applications, and this is reflected in a large fraction of the exercises. This is appreciated by most of the students. Also, from a pedagogical viewpoint, we mostly restrict our discussion to (scalar) equations of two independent variables (one space and one time, or two space variables). Most solution techniques discussed readily extend to equations of higher dimension, but this does not generally introduce new ideas that can be elaborated on at this level of course. The students are challenged enough with second order problems in two independent variables, and our main goal is to communicate key ideas in an effective way without getting too lost in a forest of calculation.

A word about the order of topics is needed, since this is a driver in the motivation for creating these notes in the first place. We start with first-order Cauchy problems (linear and semi linear) because one can tie the solution method to some geometry, the characteristic method of solution gives a straightforward recipe for students to follow, and we can immediately relate first-order problems to some interesting applications. First-order linear problems also provides a bridge to the wave equation (d'Alembert's solution) and facilitates interpretation of the solution form. So we discuss the Cauchy problem for the wave equation before we discuss it for the heat equation. Based on discussions I have had with students after the course and doing a project with other faculty members, I have included a few topics not normally included in a semester course. This includes a brief discussion of the effects of adding lower order terms to heat and wave equations, and exposing students to a higher order problem, namely the unforced and forced Bernoulli-Euler beam equation.

Finally, I never consider these Notes as static. If you have suggestions to improve the exposition, or find errors, the author would love to hear from you.

4 A Warning

Jonathan Bell
Department of Mathematics and Statistics
University of Maryland Baltimore County